**The Four Color Theorem**

In mathematics, the Four Color Theorem states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.

In this report we have discussed a brief summary of a new proof of the Four Color Theorem 8and a four-colouring algorithm found by Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas.

**Precise formulation of the theorem :**

The intuitive statement of the four color theorem, i.e. ‘that given any separation of a plane into contiguous regions, called a map, the regions can be colored using at most four colors so that no two adjacent regions have the same color’, needs to be interpreted appropriately to be correct.

A simpler statement of the theorem uses graph theory. The set of regions of a map can be represented more abstractly as an undirected graph that has a vertex for each region and an edge for every pair of regions that share a boundary segment. This graph is planar; it can be drawn in the plane without crossings by placing each vertex at an arbitrarily chosen location within the region to which it corresponds, and by drawing the edges as curves that lead without crossing within each region from the vertex location to each shared boundary point of the region. Conversely any planar graph can be formed from a map in this way. In graph-theoretic terminology, the four color theorem states that the vertices of every planar graph can be colored with at most four colors so that no two adjacent vertices receive the same color, of for short, “every planar graph is four-colorable”.

**History :**

The Four Color Problem dates back to 1852 when Francis Guthrie, while trying to color the map of countries of England notices that four colors sufficed. He asked his brother Frederick if it was true that any map can be colored using four colors in suck a way that adjacent regions (i.e. those sharing a common boundary segment, not just a point) receive different colors. Frederick Guthrie then communicated the conjecture to DeMorgan. The first printed reference is due to Cayley in 1878.

A year later the first proof by Kempe appeared; its incorrectness was pointed out by Heawood 11 years later. Another failed proof is due to Tait in 1880; a gap in the argument was pointed out by Petersen in 1891. Both failed proofs did have some value, though. Kempe discovered what became known as Kempe chains, and Tait found an equivalent formulation of the Four Color Theorem in terms of 3-edge-coloring.

The next major contribution came from Birkhoff whose work allowed Franklin in 1922 to prove that the four color conjecture is true for maps with at most 25 regions. It was also used by other mathematicians to make various forms of progress on the four color problem. We should specifically mention Heesch who developed the two main ingredients needed for the ultimate proof – reducibility and discharging. While the concept of reducibility was studied by other researchers as well, it appears that the idea of discharging, crucial for the unavoidability part of the proof, is due to Heesch, and that it was he who conjectured that a suitable development of this method would solve the Four Color Problem.

This was confirmed by Appel and Haken in 1976, when they published their proof of the Four Color Theorem.

**Why a new proof?**

There are two reasons why the Appel-Haken proof is not completely satisfactory.

* Part of the Appel-Haken proof uses a computer, and cannot be verified by hand and
* Even the part that is supposedly hand-checkable is extraordinarily complicated and tedious, and as far as they knew, no one has verified it in its entirety.

They have in fact tried to verify the Appel-Haken proof, but soon gave up. Checking the computer part would not only require a lot of programming, but also inputting the descriptions of 1476 graphs, and that was not even the most controversial part of the proof.

They decided that it would be more profitable to work out our own proof. So they did and came up with a proof and an algorithm that are described below.

**Outline of the proof :**

The basic idea of the proof is the same as Appel and Haken’s. They exhibit a set of 633 “configurations”, and prove each of them is “reducible”. This is a technical concept that implies that no configuration with this property can appear in a minimal counterexample to the Four Color Theorem. It can also be used in an algorithm, for it a reducible configuration appears in a planar graph G, then one can construct in constant time a smaller planar graph G’ such that any four-colouring of G’ can be converted to a four-colouring of G in linear time.

It has been known since 1913 that every minimal counterexample to the Four Color Theorem is an internally 6-connected triangulation. In the second part of the proof they prove that at least one of their 633 configurations appears in every internally 6-connedted planar triangulation (not necessarily a minimal counterexample to the 4CT). This is called proving unavoidability, and uses the “discharging method”, first suggested by Heesch. There their method differs from that of Appel and Haken.

**Main features of the proof :**

They confirm a conjecture of Heesch that in proving unavoidability, a reducible configuration can be found in the second neighborhood of an “overcharged” vertex; this is how we avoid “immersion” problems that were a major source of complication for Appel and Haken. Their unavoidable set has size 633 as opposed to the 1476 member set of Appel and Haken, and their discharging method uses only 32 discharging rules, instead of the 300+ of Appel and Haken. Finally, they obtain a quadratic algorithm to four-color planar graphs, an improvement over the quartic algorithm of Appel and Haken.

**Configurations :**

A near-triangulation is a non-null connected loop-less plane graph such that every finite region is a triangle. A configuration K consists of a near-triangulation G and a map g from V(G) to the integers with the following properties :

1. For every vertex v, G\v has at most two components, and if there are two, then the degree of v is g(v)-2,
2. For every vertex v, if v is not incident with the infinite region, then g(v) equals the degree of v, and otherwise g(v) is greater than the degree of v; and in either case g(v)>4,
3. K has ring-size at least 2, where the ring-size of K is defined to be the sum of g(v)-deg(v)-1, summed over all vertices v incident with the infinite region such that G\v is connected.

When drawing pictures of configurations they used a convention introduced by Heesch. The shapes of vertices indicate the value of g(v) as follows : A solid black circle means g(v)=5, a dot means g(v)=6, a hollow circle means g(v)=7, a hollow square means g(v)=8, a triangle means g(v)=9 and a pentagon means g(v)=10. Some reducible configurations are displayed **in figure below**.

Any configuration isomorphic to one of the 633 configurations is called a good configuration. Let T be a triangulation. A configuration K=(G, g) appears in T if G is an induced subgraph of T, every finite region of G is a region of T, and g(v) equals the degree of v in T for every vertex v of G. The following two statements are proved.

**Theorem 1.** If T is a minimal counterexample to the Four Color Theorem, then no good configuration appears in T.

**Theorem 2.** For every internally 6-connected triangulation T, some good configuration appears in T.

From the above two theorems it follows that no minimal counterexample exists, and so the 4CT is true. The first proof needs a computer. The second can be checked by hand in a few months, or using a computer, it can be verified in about 20 minutes.

**Discharging rules :**

Let T be an internally 6-connected triangulation. Initially, every vertex v is assigned a charge of 10(6-deg(v)). It follows from Euler’s formula that the sum of the charges of all vertices is 120; in particular, it is positive. We now redistribute the charges according to the following rules, as follows. Whenever T has a subgraph isomorphic to one of the graphs in the **figure below** satisfying the degree specifications (for a vertex v of a rule with a minus sign next to v this means that the degree of the corresponding vertex of T is at most the value specified by the shape of v, and analogously for vertices with a plus sign next to them; equality is required for vertices with no sign next to them) a charge of one (two in case of the first rule) is to be sent along the edge marked with an arrow.

This procedure defines a new set of charges with the same total sum. Since the total sum is positive, there is a vertex v in T whose new charge is positive. We show that a good configuration appears in the second neighborhood of v.

If the degree of v is at most 6 of at least 12, then this can be seen fairly easily by a direct argument. For the remaining cases, however, the proofs are much more complicated. Therefore, they have written the proofs in a formal language so that they can be verified by a computer. Each individual step of these proofs is human-verifiable, but the proofs themselves are not really checkable by hand, because of their length.